

Computational Complexity of Testing Proportional Justified Representation

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Abstract

We consider a committee voting setting in which each voter approves of a subset of candidates and based on the approvals, a target number of candidates are selected. Aziz et al. (2015) proposed two representation axioms called justified representation and extended justified representation. Whereas the former can be tested as well as achieved in polynomial time, the latter property is coNP-complete to test and no polynomial-time algorithm is known to achieve it. Interestingly, Sánchez-Fernández et al. (2016) proposed an intermediate property called proportional justified representation that admits a polynomial-time algorithm to achieve. The complexity of testing proportional justified representation has remained an open problem. In this paper, we settle the complexity by proving that testing proportional justified representation is coNP-complete. We complement the complexity result by showing that the problem admits efficient algorithms if any of the following parameters are bounded: (1) number of voters (2) number of candidates (3) maximum number of candidates approved by a voter (4) maximum number of voters approving a given candidate.

Keywords: Social choice theory, committee voting, multi-winner voting, approval voting, computational complexity

JEL: C63, C70, C71, and C78

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1. Introduction

We consider a committee voting setting in which each voter approves of a subset of candidates and based on the approvals, a target k number of candidates are selected. The setting has been referred to as approval-based multiwinner voting or committee voting with approvals. The setting has inspired a number of natural voting rules [8, 5, 9, 3, 13]. Many of the voting rules attempt to satisfy some notion of representation. However it has been far from clear what axiom capture the representation requirements.

Aziz et al. [1, 2] proposed two compelling representation axioms called *justified representation (JR)* and *extended justified representation (EJR)*. Whereas the former can be tested as well as achieved in polynomial time, the latter property is coNP-complete to test and no polynomial-time algorithm is known to achieve it. Interestingly, Sánchez-Fernández et al. [11] proposed an intermediate property called *proportional justified representation (PJR)* that admits a polynomial-time algorithm to achieve. The idea behind all the three properties is that a cohesive and large enough group deserves sufficient number of approved candidates in the winning set of candidates. Sánchez-Fernández et al. [11] argued that although *EJR* is a stronger property than *PJR*, *PJR* is more reasonable because it is compatible with a property called perfect representation.

Proportional justified representation (*PJR*) has been examined in subsequent papers [6, 12, 10]. Despite the flurry of work on the property, the complexity of testing proportional justified representation has remained an open problem. Sánchez-Fernández et al. [10] state that “*we do not know what is the complexity of checking whether a given committee provides PJR*”. In a talk “Approval Voting, representation, & Liquid Democracy” at the *Workshop on Future Directions in Computational Social Choice, Hungary* in November 2016, Markus Brill also mentioned the problem as an interesting open problem.¹ The problem is especially important if one wants to test whether a status quo outcome or the outcome of some other rule or negotiation process satisfies *PJR*. Previously, Aziz et al. [4] studied the complexity of testing Pareto optimality of a committee.

In this paper, we settle the complexity of testing *PJR* by proving that the problem is coNP-complete. We complement the complexity result by showing that the problem admits efficient algorithms if any of the following parameters are bounded: (1) n (number of voters) (2) m (number of candidates) (3) a (maximum number of candidates approved by a voter) (4) d (maximum number of voters approving a given candidate). For the first two parameters, we show that the problem is *FPT (fixed-parameter tractable)*, i.e, there exists an FPT algorithm that solves the problem in $f(k) \cdot \text{poly}(|I|)$ time, where k is the parameter and f is some computable function and poly is a polynomial both independent of problem instance I .

¹http://econ.core.hu/file/download/future_markus.pdf

Our results are summarized in Table 1.

Parameter	Complexity	Reference
—	coNP-complete	Th. 1
n : # voters	in FPT: $O(2^n mn)$	Th. 2
m : # candidates	in FPT: $O(2^m m^3 n)$	Th. 3
$a : \max_{i \in N} A_i $	in P for constant a : $O(m^{a+1} m^2 n)$	Th. 4
$d : \max_{c \in C} \{i \in N : c \in A_i\} $	in P for constant d : $O(n^d d n m^2)$	Th. 5

Table 1: Complexity of testing PJR

2. Approval-based Committee Voting and Representation Properties

We consider a social choice setting with a set $N = \{1, \dots, n\}$ of voters and a set C of m candidates. Each voter $i \in N$ submits an approval ballot $A_i \subseteq C$, which represents the subset of candidates that she approves of. We refer to the list $\vec{A} = (A_1, \dots, A_n)$ of approval ballots as the *ballot profile*. We will consider *approval-based multi-winner voting rules* that take as input a tuple (N, C, \vec{A}, k) , where k is a positive integer that satisfies $k \leq m$, and return a subset $W \subseteq C$ of size k , which we call the *winning set*, or *committee*.

We now summarize the main representation properties proposed in the literature.

Definition 1 (Justified representation (JR)). *Given a ballot profile $\vec{A} = (A_1, \dots, A_n)$ over a candidate set C and a target committee size k , we say that a set of candidates W of size $|W| = k$ provides justified representation for (\vec{A}, k) if*

$$\forall X \subseteq N : |X| \geq \frac{n}{k} \text{ and } |\cap_{i \in X} A_i| \geq 1 \implies (|W \cap (\cup_{i \in X} A_i)| \geq 1)$$

The rationale behind this definition is that if k candidates are to be selected, then, intuitively, each group of $\frac{n}{k}$ voters “deserves” a representative. Therefore, a set of $\frac{n}{k}$ voters that have at least one candidate in common should not be completely unrepresented.

Definition 2 (Extended justified representation (EJR)). *Given a ballot profile (A_1, \dots, A_n) over a candidate set C , a target committee size k , $k \leq m$, we say that a set of candidates W , $|W| = k$, satisfies ℓ -extended justified representation for (\vec{A}, k) and integer ℓ if*

$$\forall X \subseteq N : |X| \geq \ell \frac{n}{k} \text{ and } |\cap_{i \in X} A_i| \geq \ell \implies (\exists i \in X : |W \cap A_i| \geq \ell).$$

We say that W satisfies extended justified representation for (\vec{A}, k) if it satisfies ℓ -extended justified representation for (\vec{A}, k) and all integers $\ell \leq k$.

Sánchez-Fernández et al. [11] came up with the notion of *proportional justified representation (PJR)*, which can be seen as an alternative to *EJR*.

Definition 3 (Proportional Justified Representation (*PJR*)). *Given a ballot profile (A_1, \dots, A_n) over a candidate set C , a target committee size k , $k \leq m$, and integer ℓ we say that a set of candidates W , $|W| = k$, satisfies ℓ -proportional justified representation for (\vec{A}, k) if*

$$\forall X \subseteq N : |X| \geq \ell \frac{n}{k} \text{ and } |\cap_{i \in X} A_i| \geq \ell \implies (|W \cap (\cup_{i \in X} A_i)| \geq \ell)$$

We say that W satisfies proportional justified representation for (\vec{A}, k) if it satisfies ℓ -proportional justified representation for (\vec{A}, k) and all integers $\ell \leq k$.

It is easy to observe that *EJR* implies *PJR* which implies *JR*.

3. Results

We first prove that testing *PJR* is coNP-complete. The proof involves a similar type of reduction as the one used by Aziz et al. [1, 2] to prove that testing *EJR* is coNP-complete.

Theorem 1. *Given a ballot profile \vec{A} , a target committee size k , and a committee W , $|W| = k$, it is coNP-complete to check whether W provides *PJR* for (\vec{A}, k) .*

Proof. It is easy to see that this problem is in coNP. A set of voters $X \subset N$ such that $|X| \geq \ell \frac{n}{k}$, $|\cap_{i \in X} A_i| \geq \ell$ and $|W \cap (\cup_{i \in X} A_i)| < \ell$ is a certificate that W does not satisfy *PJR*.

To prove coNP-completeness, we reduce the classic BALANCED BICLIQUE problem ([GT24] in Garey and Johnson 1979) to the complement of our problem. An instance of BALANCED BICLIQUE is given by a bipartite graph (L, R, E) with parts L and R and edge set E , and an integer ℓ ; it is a “yes”-instance if we can pick subsets of vertices $L' \subseteq L$ and $R' \subseteq R$ so that $|L'| = |R'| = \ell$ and $(u, v) \in E$ for each $u \in L', v \in R'$; otherwise, it is a “no”-instance.

Given an instance $\langle (L, R, E), \ell \rangle$ of BALANCED BICLIQUE with $R = \{v_1, \dots, v_s\}$, we create an instance of our problem as follows. Assume without loss of generality that $s \geq 3$, $\ell \geq 3$. We construct 3 pairwise disjoint sets of candidates C_0, C_1 and C_2 , so that $C_0 = L$, $|C_1| = \ell - 1$, $|C_2| = s\ell + \ell - 3s + (\ell - 2)$, and set $C = C_0 \cup C_1 \cup C_2$. We then construct 3 sets of voters N_0, N_1, N_2 , so that $N_0 = \{1, \dots, s\}$, $|N_1| = \ell(s - 1) + \ell$, $|N_2| = s\ell + \ell - 3s + (\ell - 2)$ (note that $|N_2| \geq (\ell - 1)$ as we assume that $\ell \geq 3$). For each $i \in N_0$ we set $A_i = \{u_j \mid (u_j, v_i) \in E\}$, and for each $i \in N_1$ we set $A_i = C_0 \cup C_1$. The candidates in C_2 are matched to voters in N_2 : each voter in N_2 approves exactly one candidate in C_2 , and each candidate in C_2 is approved by exactly one voter in N_2 . Denote the resulting list of ballots by \vec{A} . Finally, we set $k = 2\ell - 2$, and let $W = C_1 \cup X$, where X is a subset of C_2 with $|X| = \ell - 1$. Note that the number

of voters n is given by $s + \ell(s - 1) + \ell + s\ell + \ell - 3s + (\ell - 2) = 2(s + 1)(\ell - 1)$, so $\frac{n}{k} = s + 1$.

Suppose first that we started with a “yes”-instance of BALANCED BICLIQUE, and let (L', R') be the respective ℓ -by- ℓ biclique. Let $C^* = L'$ and $N^* = R' \cup N_1$. Then $|N^*| = \ell(s + 1) = \ell \frac{n}{k}$, all voters in N^* approve all candidates in C^* , $|C^*| = \ell$, but all voters in N^* together are only represented by $\ell - 1$ candidates in W . Hence, W fails to provide ℓ -proportional justified representation for (\vec{A}, k) .

Conversely, suppose that W fails to provide PJR for (\vec{A}, k) . That is, there exists a value $j > 0$, a set N^* of $j(s + 1)$ voters and a set C^* of j candidates so that all voters in N^* approve of all candidates in C^* , but all voters in N^* together are only represented by less than j candidates in W . Note that, since $s > 1$ and $j \geq 1$, we have $N^* \cap N_2 = \emptyset$. Further, since $|N^*| = j(s + 1) \geq s + 1$ and $|N_0| = s$, it follows that N^* contains one voter from N_1 . So, all voters in N^* together are represented by exactly $\ell - 1$ candidates in W . This implies that $j \geq \ell$. As $N^* = j(s + 1) \geq \ell(s + 1)$, it follows that $|N^* \cap N_0| \geq \ell$. Since N^* contains voters from both N_0 and N_1 , it follows that $C^* \subseteq C_0$. Thus, there are at least ℓ voters in $N^* \cap N_0$ who approve the same $j \geq \ell$ candidates in C_0 ; any set of ℓ such voters and ℓ such candidates corresponds to an ℓ -by- ℓ biclique in the input graph. \square

Note that although there is a polynomial-time algorithm to compute a committee that achieves PJR [12], we have proved that checking whether any arbitrary committee achieves PJR is coNP-complete. We complement the negative computational result by showing that testing PJR is computationally tractable if one of the following parameters is bounded.

- $m = |C|$
- $n = |N|$
- $a = \max_{i \in N} |A_i|$ (maximum size of approval sets).
- $d = \max_{c \in C} |\{i \in N : c \in A_i\}|$ (maximum number of approvals of an alternative).

We first observe that testing PJR is in FPT with parameter n .

Theorem 2. *Testing PJR is in FPT with parameter n and takes time at most $O(2^n mn)$.*

Proof. Suppose we want to check whether $W \subset C$ satisfies PJR . If n is bounded then one can simply brute force all the possible violating sets $X \subseteq N$ of voters and check that if $|X| \geq \ell \frac{n}{k}$ and $|\cap_{i \in X} A_i| \geq \ell$ then it must be that $|W \cap (\cup_{i \in X} A_i)| \geq \ell$. \square

Next, we prove that testing PJR is in FPT with parameter m .

Theorem 3. *Testing PJR is in FPT with parameter m and takes time at most $O(2^m m^3 n)$.*

Proof. Suppose we want to check whether $W \subset C$ satisfies PJR . Note that it is sufficient to show that testing ℓ - PJR is in FPT with parameter m . Note that $\ell \leq |X|k/n$ for all $X \subset N$. Since $|X| \leq n$, $\ell \leq k \leq m$.

We go through all the subsets $S \in 2^C$ of size ℓ . Each set S is viewed as the intersection of possible objecting/deviating set of voters. For each S , we find the corresponding set of voters X_S as follows:

$$X_S = \{i \in N : A_i \subseteq S\}.$$

We return no (i.e., W does not satisfy ℓ - PJR) if $|X_S| \geq \ell \frac{n}{k}$, $|\cap_{i \in X} A_i| \geq \ell$ but $|W \cap (\cup_{i \in X} A_i)| < \ell$. If we do not return no for any S , in that case we return yes (i.e., W satisfies ℓ - PJR).

We now argue that it takes at most $O(2^m m^3 n)$ operations to test PJR . To check ℓ - PJR , we go through 2^m sets. For each set S , we find X_S which takes $m^2 n$ steps. After that we find $\cup_{i \in X} A_i$ which takes an additional mn operations. Hence it takes $O(2^m m^2 n)$ operations to test ℓ - PJR and it takes $O(2^m m^3 n)$ operations to test PJR . \square

If $a = \max_{i \in N} |A_i|$ is bounded, then PJR can be tested in polynomial time.

Theorem 4. *If a is bounded, testing PJR is solvable in polynomial time $O(m^{a+1} m^2 n)$.*

Proof. Suppose we want to check whether $W \subset C$ satisfies PJR . Note that it is sufficient to show that testing ℓ - PJR is polynomial-time solvable for each ℓ if a is bounded. Note that we only need to consider $\ell \leq a$ because the maximum size of intersection of any set of approval sets is at most a which means that $|\cap_{i \in X} A_i| \leq a$. For ℓ larger than a , ℓ - PJR is trivially satisfied.

We now describe the algorithm to test ℓ - PJR for all $\ell \leq a$. We go through all the subsets $S \in 2^C$ of size $\ell \leq a$. There are at most $\binom{m}{\ell} = \frac{m!}{(m-\ell)!(\ell)!}$ such sets. Since $\ell \leq a$ and a is bounded, it implies that a is constant as well and hence there at most m^a different subsets to be considered.

Each set S is viewed as the intersection of possible objecting /deviating set of voters. For each S , we find the corresponding set of voters X_S as follows:

$$X_S = \{i \in N : A_i \subseteq S\}.$$

We return no (i.e., W does not satisfy ℓ - PJR) if $|X_S| \geq \ell \frac{n}{k}$, $|\cap_{i \in X} A_i| \geq \ell$ but $|W \cap (\cup_{i \in X} A_i)| < \ell$. If we do not return no for any S , in that case we return yes (i.e., W satisfies ℓ - PJR).

We now argue that it takes at most $O(m^a m^3 n)$ operations to test PJR . To check ℓ - PJR , we go through at most m^a sets. For each set S , we find X_S which takes $m^2 n$ steps. After that we find $\cup_{i \in X} A_i$ which takes an additional mn operations. Hence it takes $O(m^a m^2 n)$ operations to test ℓ - PJR and it takes $O(m^{a+1} m^2 n)$ operations to test PJR . \square

Finally, we show that if $d = \max_{c \in C} |\{i \in N : c \in A_i\}|$ is bounded, then PJR can be tested in polynomial time.

Theorem 5. *If d is bounded, testing PJR is solvable in polynomial time $O(n^d dnm^2)$.*

Proof. Suppose we want to check whether $W \subset C$ satisfies PJR . Note that for any deviating/objecting set of voters X , $|\cap_{i \in X} A_i| \leq d$. The reason is that any candidate c is approved by at most d voters. Hence in order to check for a violation of ℓ - PJR , we just need to check for sets of voters of size at most d . There are at most $\binom{n}{d} \leq n^d$ such set of voters. For each such coalition of voters, we just need to check for ℓ - PJR for $\ell = 1, \dots, d/k$. So $\ell \leq d$. For each X among the at most n^d sets of voters, we need to test for ℓ - PJR which requires us to compute $|\cap_{i \in X} A_i|$, $|W \cap (\cup_{i \in X} A_i)|$, and $|X|$. Hence testing ℓ - PJR of W takes $O(n^d nm^2)$ time and testing PJR takes time $O(n^d dnm^2)$. \square

In this paper, we examined the complexity of testing PJR , an interesting new axiom in committee voting. The arguments for all of our positive algorithmic results also hold for testing EJR rather than PJR .

References

- [1] Aziz, H., Brill, M., Conitzer, V., Elkind, E., Freeman, R., Walsh, T., 2015. Justified representation in approval-based committee voting. In: Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI). AAAI Press, pp. 784–790.
- [2] Aziz, H., Brill, M., Conitzer, V., Elkind, E., Freeman, R., Walsh, T., 2017. Justified representation in approval-based committee voting. Social Choice and Welfare.
- [3] Aziz, H., Gaspers, S., Gudmundsson, J., Mackenzie, S., Mattei, N., Walsh, T., 2015. Computational aspects of multi-winner approval voting. In: Proceedings of the 14th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS). IFAAMAS, pp. 107–115.
- [4] Aziz, H., Lang, J., Monnot, J., 2016. Computing Pareto Optimal Committees. In: Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI). pp. 60–66.
- [5] Brams, S. J., Fishburn, P. C., 2007. Approval Voting, 2nd Edition. Springer-Verlag.
- [6] Brill, M., Freeman, R., Janson, S., Lackner, M., 2017. Phragmén’s voting methods and justified representation. In: Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI). AAAI Press, forthcoming.
- [7] Garey, M. R., Johnson, D. S., 1979. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman.
- [8] Kilgour, D. M., 2010. Approval balloting for multi-winner elections. In: Laslier, J.-F., Sanver, M. R. (Eds.), Handbook on Approval Voting. Springer, Ch. 6, pp. 105–124.

- [9] LeGrand, R., Markakis, E., Mehta, A., 2007. Some results on approximating the minimax solution in approval voting. In: Proceedings of the 6th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS). IFAAMAS, pp. 1193–1195.
- [10] Sánchez-Fernández, L., Elkind, E., Lackner, M., Fernández, N., Fisteus, J. A., Basanta Val, P., Skowron, P., 2017. Proportional justified representation. In: Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI). AAAI Press, forthcoming.
- [11] Sánchez-Fernández, L., Fernández, N., Fisteus, J. A., Basanta Val, P., 2016. Some notes on justified representation. In: Proceedings of the 10th Multi-disciplinary Workshop on Advances in Preference Handling (MPREF).
- [12] Sánchez-Fernández, L., Fernández, N., Fisteus, L. A., 2016. Fully open extensions to the D’Hondt method. Tech. Rep. arXiv:1609.05370 [cs.GT], arXiv.org.
- [13] Skowron, P. K., Faliszewski, P., Lang, J., 2016. Finding a collective set of items: From proportional multirepresentation to group recommendation. *Artificial Intelligence* 241, 191–216.